Effect of the DDM3Y Nuclear Equation of State on the r-mode instability of neutron stars

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The density dependent M3Y interaction potential is used for isoscalar and isovector part:

\[ v_{00}(s, \rho, \varepsilon^{\text{kin}}) = t_{00}^{\text{M3Y}}(s, \varepsilon^{\text{kin}}) g(\rho), \quad v_{01}(s, \rho, \varepsilon^{\text{kin}}) = t_{01}^{\text{M3Y}}(s, \varepsilon^{\text{kin}}) g(\rho) \]

Isoscalar \( t_{00}^{\text{M3Y}} \) and isovector \( t_{01}^{\text{M3Y}} \) components of M3Y interaction supplemented by zero range potential representing the single nucleon exchange term are given as

\[
t_{00}^{\text{M3Y}} = 7999 \frac{\exp(-4s)}{4s} - 2134 \frac{\exp(-2.5s)}{2.5s} - 276(1 - \alpha \varepsilon^{\text{kin}}) \delta(s)
\]

\[
t_{01}^{\text{M3Y}} = -4886 \frac{\exp(-4s)}{4s} + 1176 \frac{\exp(-2.5s)}{2.5s} + 228(1 - \alpha \varepsilon^{\text{kin}}) \delta(s)
\]

where the energy dependence parameter \( \alpha = 0.005 \text{ MeV}^{-1} \). Strengths of the Yukawas are extracted by fitting its matrix elements in an oscillator basis to those elements of G-matrix obtained with Reid-Elliott soft core NN interaction. The density dependence of the form \( g(\rho) = C(1 - b \rho^{2/3}) \) accounts Pauli blocking effects and higher order exchange effects.
Isoscalar and isovector components of the effective interaction

The central part of the effective interaction between two nucleons 1 and 2 can be written as:

\[ v_{12}(s) = v_{00}(s) + v_{01}(s)\tau_1\cdot\tau_2 + v_{10}(s)\sigma_1\cdot\sigma_2 - v_{11}(s)(\sigma_1\cdot\sigma_2)(\tau_1\cdot\tau_2) \]

where \( \tau_1, \tau_2 \) are the isospins, \( \sigma_1, \sigma_2 \) are the spins and \( s \) is the distance between nucleons 1, 2.

For spin symmetric nucleons \( v_{10} \) and \( v_{11} \) do not contribute.

Z-component of Isospins (\( I_z \)) of proton and neutron are +1 and -1. \( \tau_n\cdot\tau_p = \tau_p\cdot\tau_n = -1 \) and \( \tau_n\cdot\tau_n = \tau_p\cdot\tau_p = +1 \).

For SNM only the first term, the isoscalar term, contributes. For IANM the first two terms the isoscalar and the isovector (Lane) terms contribute.

The n-n and p-p interactions are \( v_{nn} = v_{pp} = v_{00} + v_{01} \)

The n-p and p-n interactions are \( v_{np} = v_{pn} = v_{00} - v_{01} \)
Symmetric and isospin asymmetric nuclear matter calculations

For a single neutron interacting with rest of nuclear matter with isospin asymmetry \( X = (\rho_n - \rho_p)/(\rho_n + \rho_p) \), the interaction energy per unit volume at \( s \) is:

\[
\rho_n v_{nn}(s) + \rho_p v_{np}(s) = \rho_n [v_{00}(s) + v_{01}(s)] + \rho_p [v_{00}(s) - v_{01}(s)] \\
= \rho [v_{00}(s) + Xv_{01}(s)]
\]

Similarly, for the case of proton the interaction energy per unit volume

\[
\rho_n v_{pn}(s) + \rho_p v_{pp}(s) = \rho [v_{00}(s) - Xv_{01}(s)]
\]

- Asymmetric nuclear EOS can be applied to study the pure neutron matter with isospin asymmetry \( X = 1 \).

- The bulk properties of neutron matter such as the nuclear incompressibility \( (K_o) \), the energy density, the pressure and the velocity of sound in nuclear medium can be used to study the cold compact stellar object like neutron star.
Kinetic & Potential energy of a nucleon in symmetric nuclear matter

K.E./A = \int_0^{k_F} \frac{h^2 k^2}{2m} \frac{4d^3 p}{h^3} = \frac{3}{5} \frac{h^2 k_F^2}{2m} \int_0^{k_F} \frac{4d^3 p}{h^3}

P.E./A = \frac{1}{2} \int_0^{k_F} \int_0^{k_F} \int_0^{k_F} v_{00}(s) \frac{4d^3 p_2}{h^3} d^3 s \frac{4d^3 p_1}{h^3} d^3 r = \frac{1}{2} \int_0^{k_F} v_{00}(s) \frac{4d^3 p_2}{h^3} d^3 s

= \frac{1}{2} \int v_{00}(s) d^3 s \int_0^{k_F} \frac{4d^3 p_2}{h^3} = \frac{1}{2} \rho \int_0^{\infty} v_{00}(s) d^3 s = \frac{1}{2} \rho g(\rho) J_{v00}

where \( J_{v00} = \int_0^{\infty} t_{00}^{MY}(s, \epsilon^{\text{kin}}) d^3 s \)
Calculations for Symmetric Nuclear Matter

\[ \varepsilon = \frac{3\hbar^2k_F^2}{10m} + \frac{C}{2}(1 - \beta \rho^n)\rho J_{v00} \]

\[ \frac{\partial \varepsilon}{\partial \rho} = \frac{\hbar^2k_F^2}{5m\rho} + \frac{C}{2}[1 - (n+1)\beta \rho^n)]J_{v00} - \alpha J_{00}C(1 - \beta \rho^n)\frac{\hbar^2k_F^2}{10m} \]

where \( \alpha = 0.005 \text{ MeV}^{-1} \) and \( J_{00} = -276 \text{ MeV} \)

Using: 1. \( \frac{\partial \varepsilon}{\partial \rho} = 0 \) at \( \rho = \rho_0 \) 2. \( \varepsilon = \varepsilon_0 \) at \( \rho = \rho_0 \)

where saturation energy per nucleon = \( \varepsilon_0 \)

and the saturation density = \( \rho_0 \)

\[ \beta = \frac{\rho_0^{-n}(1 - p + q - 3q/p)}{3n + 1 - (n+1)p + q - 3q/p} \]

\[ C = -\frac{2\hbar^2k_{F0}^2}{5m\rho_0 J_{v00}^0} \frac{1}{1 - (n+1)\beta \rho_0^n - q(1 - \beta \rho_0^n)/p} \]

where \( p = \frac{10m\varepsilon_0}{\hbar^2k_{F0}^2} \), \( q = \frac{2\alpha \varepsilon_0 J_{00}}{J_{v00}^0} \) and \( J_{v00}^0 = J_{v00} \) at \( \varepsilon^{kin} = \varepsilon_0^{kin} \)
The isospin asymmetry parameter $X = (\rho_n - \rho_p) / (\rho_n + \rho_p)$ with density $\rho = \rho_n + \rho_p$.

For $X = 0$

Symmetric Nuclear Matter (SNM), $X = 0$, contains same no. of $n$ and $p$ ($\rho_n = \rho_p$).

Towards asymmetry
Change $X$ from 0

Isospin Asymmetric Nuclear Matter (IANM), $X \neq 0$, contains different no. of $n$ and $p$ ($\rho_n \neq \rho_p$).

Range of $X$: $-1 \leq X \leq 1$
Equation of State for IANM

Assuming interacting Fermi gas of neutrons and protons, the kinetic energy per nucleon $\varepsilon_{\text{kin}}$ turns out to be

$$\varepsilon_{\text{kin}} = \frac{3\hbar^2 k_F^2}{10m} F(X)$$

where $F(X) = \frac{1}{2}[(1 + X)^{5/3} + (1 - X)^{5/3}]$

The EoS for IANM:

$$\frac{E}{A} = \varepsilon = \frac{3\hbar^2 k_F^2}{10m} F(X) + \frac{C}{2} (1 - \beta \rho^n) \rho J_v$$

where $J_v = J_{v00} + X^2 J_{v01} = \int_0^\infty (t_{00}^{M3Y} + t_{01}^{M3Y} X^2) d^3s$

considering energy variation of zero range potential to vary with $\varepsilon_{\text{kin}}$. 
$E/A = \varepsilon$ of NM with different X as functions of $\rho/\rho_0$ for present calc.

$\varepsilon_{\text{SNM}}$ is negative up to $2\rho_0$ (Bound)

$\varepsilon_{\text{PNM}} > 0$ always unbound by nuclear interaction

$\varepsilon_{\text{SNM}} = 15.26 \pm 0.52$ MeV < 0 for SNM
The pressure $P$ of SNM as a function of $\rho/\rho_0$ is consistent with experimental flow data for SNM.

3 curves of this work for $\epsilon = 15.26 \pm 0.52$ MeV


Akmal et al.

RMF NL3
The pressure $P$ of PNM as a function of $\rho/\rho_0$ is consistent with experimental flow data for PNM with weak (soft NM) and strong (stiff NM) $\rho$-dependence.
$K_\tau$ versus $K_0$ ($K_{\text{inf}}$) for the present calculation using DDM3Y effective interaction and comparison with the other predictions. The dotted rectangular region encompasses the values of $K_0 = 250 - 270$ MeV [1] and $K_\tau = -370 \pm 120$ MeV [5].
\[ K_t = K_{\text{sym}} - 6L - \left( Q_0 / K_0 \right) L \]

\[ K_{\text{sym}} = 9 \rho_0^2 \left[ \frac{\partial^2 E_{\text{sym}}}{\partial \rho^2} \right]_{\rho=\rho_0} \]

\[ L = 3 \rho_0 \left[ \frac{\partial E_{\text{sym}}}{\partial \rho} \right]_{\rho=\rho_0} \]

\[ Q_0 = 27 \rho_0^3 \left[ \frac{\partial^3 \varepsilon_{\text{SNM}}}{\partial \rho^3} \right]_{\rho=\rho_0} \]

**Isobaric incompressibility**

\[ K_0(X) = K_0 + K_t X^2 + O(X^4) \]

<table>
<thead>
<tr>
<th>Model</th>
<th>$K_0(X=0)$</th>
<th>$E_{\text{sym}}(\rho_0)$</th>
<th>$L$</th>
<th>$K_{\text{sym}}$</th>
<th>$K_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>This work</td>
<td>274.7±7.4</td>
<td>30.71±0.26</td>
<td>45.11±0.02</td>
<td>-183.7±3.6</td>
<td>-408.97±3.01</td>
</tr>
<tr>
<td>NL3</td>
<td>271.56</td>
<td>37.29</td>
<td>118.20</td>
<td>100.90</td>
<td>-697.36</td>
</tr>
<tr>
<td>DDME1</td>
<td>244.04</td>
<td>33.10</td>
<td>54.51</td>
<td>-103.00</td>
<td>-559.49</td>
</tr>
<tr>
<td>DDME2</td>
<td>250.28</td>
<td>33.10</td>
<td>50.44</td>
<td>-89.05</td>
<td>-541.58</td>
</tr>
<tr>
<td>FSUGold</td>
<td>230.00</td>
<td>32.59</td>
<td>60.50</td>
<td>-51.30</td>
<td>-276.77</td>
</tr>
</tbody>
</table>

[2]. From different experimental studies
[5]. Lie-Wen Chen et al. PRC 80 (2009) 014322
The nuclear symmetry energy $E_{\text{sym}}(\rho)$ represents a penalty levied on the system as it departs from the symmetric limit of equal number of protons and neutrons and can be defined as the energy required per nucleon to change the symmetric nuclear matter to pure neutron matter and hence

$$E_{\text{sym}}(\rho) = \varepsilon(\rho, X = 1) - \varepsilon(\rho, X = 0)$$

From the most physical definition of the nuclear symmetry energy as defined above, the resent calculation gives a value of $E_{\text{sym}}(\rho_0) = 30.71 \pm 0.26$ MeV that is consistent with the empirical value extracted by fitting the droplet model to the measured atomic mass excesses using maximum likelihood estimator method.

If we use the alternative definition

$$E_{\text{sym}}(\rho) = \frac{1}{2} \left[ \frac{\partial^2 \varepsilon(\rho, X)}{\partial X^2} \right]_{X = 0}$$

the value of nuclear symmetry energy remains almost same $30.03 \pm 0.26$ MeV.

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**Coefficients of the liquid droplet model mass formula extracted from atomic mass excesses**

<table>
<thead>
<tr>
<th>$a_u$</th>
<th>$a_x$</th>
<th>$a_c$</th>
<th>$S_v$</th>
<th>$S_z$</th>
<th>$a_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MeV</td>
<td>MeV</td>
<td>MeV</td>
<td>MeV</td>
<td>MeV</td>
<td>MeV</td>
</tr>
<tr>
<td>15.500 ± 0.00012$^a$</td>
<td>17.480 ± 0.00025</td>
<td>0.689 ± 0.0002</td>
<td>30.048 ± 0.0042</td>
<td>16.674 ± 0.0071</td>
<td>10.246 ± 0.00048</td>
</tr>
<tr>
<td>15.465 ± 0.00012$^b$</td>
<td>17.394 ± 0.00025</td>
<td>0.686 ± 0.00002</td>
<td>30.130 ± 0.0043</td>
<td>16.317 ± 0.0067</td>
<td>10.273 ± 0.00049</td>
</tr>
</tbody>
</table>

$^a$ Using only the experimentally measured 2228 atomic mass excesses. $\mu_{th} = 0.029$ and $\sigma_{th} = 2.880$.

$^b$ Using both measured 2228 + extrapolated 951 atomic mass excesses. $\mu_{th} = 0.040$ and $\sigma_{th} = 2.960$. 
R-Mode Instability

• It is a rotational instability of a neutron star due to fluid velocity perturbation whose restoring force is the Coriolis force.
• The r-mode frequency has different signs in the co-rotating and inertial frames. They are retrograde in co-rotating frame and prograde in inertial frame.
• They are destabilized by the Chandrasekhar-Friedman-Schutz (CFS) mechanism and are unstable because of the emission of gravitational waves. The gravitational radiation that the r-modes emit comes from their time-dependent mass currents.
• The instability in the r-modes is damped by viscosity. For the instability to be relevant, it must grow faster than it is damped out by the viscosity. So, the timescale for gravitationally driven instability needs to be sufficiently short to the viscous damping timescale.
R-Mode Instability

- The timescale associated with the different process involves the actual physical parameters of the neutron star. In computing these physical parameters, the role of nuclear physics comes into the picture, where one gets a platform to constrain the uncertainties existing in the nuclear EoS.

- Using the core-crust transition point using the DDM3Y effective NN interaction we calculate the r-mode instability windows of NS with a rigid crust in the slow rotation approximation.

- We consider the temperature of NS less then equal to $10^9$ K so that the dominant dissipative mechanism is the shear viscosity along the crust-core boundary layer.
DISSIPATIVE TIMESCALES AND STABILITY OF THE R-MODES

The amplitude of r modes evolves with time dependence as

\[ e^{i \omega (t - t_0) / \tau} \]

where,

\[
\frac{1}{\tau(\Omega, T)} = \frac{1}{\tau_{GR}(\Omega, T)} + \frac{1}{\tau_{BV}(\Omega, T)} + \frac{1}{\tau_{SV}(\Omega, T)}
\]

\[
\frac{1}{\tau_{GR}} = -\frac{32\pi G \Omega^{2l+2}}{c^{2l+3}} \frac{(l-1)^{2l}}{[(2l+1)!!]^2} \left( \frac{l+2}{l+1} \right)^{(2l+2)}
\times \int_0^{R_c} \rho(r) r^{2l+2} dr,
\]

\[
\frac{1}{\tau_{SV}} = \left[ \frac{1}{2} \frac{2^{l+3/2}(l+1)!}{2\Omega l(2l+1)!! l!} \sqrt{\frac{2\Omega R_c^2 \rho_c}{\eta_c}} \right]^{-1}
\times \left[ \int_0^{R_c} \frac{\rho(r)}{\rho_c} \left( \frac{r}{R_c} \right)^{2l+2} dr \right]^{-1},
\]
DISSIPATIVE TIMESCALES AND STABILITY OF THE R-MODES

\[
\frac{1}{\tau_{SV}} = \frac{1}{\tau_{ee}} + \frac{1}{\tau_{nn}},
\]

\[
\eta_{SV}^{ee} = 6 \times 10^6 \rho^2 T^{-2} \quad (\text{g cm}^{-1} \text{s}^{-1}),
\]

\[
\eta_{SV}^{nn} = 347 \rho^{9/4} T^{-2} \quad (\text{g cm}^{-1} \text{s}^{-1}),
\]

\[
\frac{1}{\tau(\Omega, T)} = \frac{1}{\tilde{\tau}_{GR}} \left( \frac{\Omega}{\Omega_0} \right)^{2/11} + \frac{1}{\tilde{\tau}_{SV}} \left( \frac{\Omega}{\Omega_0} \right)^{1/2} \left( \frac{10^8 \text{K}}{T} \right).
\]

where \( \Omega_0 = \sqrt{\pi G \tilde{\rho}} \) and \( \tilde{\rho} = 3M/4\pi R^3 \)

Critical angular velocity:

\[
\left( \frac{\Omega_c}{\Omega_0} \right) = \left( -\frac{\tilde{\tau}_{GR}}{\tilde{\tau}_{SV}} \right)^{2/11} \left( \frac{10^8 \text{K}}{T} \right)^{2/11}.
\]
DISSIPATIVE TIMESCALES AND STABILITY OF THE R-MODES

Kepler frequency: \(\Omega_K \approx \frac{2}{3} \Omega_0\).

Critical Temperature:

\[
\frac{T_c}{10^8 \text{K}} = \left(\frac{\Omega_0}{\Omega_c}\right)^{11/2} \left(\frac{\tau_{GR}}{\tau_{SV}}\right) \approx (3/2)^{11/2} \left(\frac{\tau_{GR}}{\tau_{SV}}\right).
\]

\[
\left(\frac{\Omega_c}{\Omega_0}\right) = \frac{\Omega_K}{\Omega_0} \left(\frac{T_c}{T}\right)^{2/11} \approx (2/3) \left(\frac{T_c}{T}\right)^{2/11}.
\]
SPIN DOWN DUE TO R-MODE INSTABILITY

- R-mode instability causes gravitational radiation to emit which spins down the neutron star. The angular velocity evolution is given by:

\[
\frac{d\Omega}{dt} = \frac{2\Omega}{\tau_{GR}} \frac{\alpha_r^2 Q}{1 - \alpha_r^2 Q}
\]

\[
Q = \frac{3\tilde{J}}{2\tilde{I}}
\]

\[
\tilde{J} = \frac{1}{MR^4} \int_0^R \rho(r)r^6 dr
\]

\[
\tilde{I} = \frac{8\pi}{3MR^2} \int_0^R \rho(r)r^4 dr.
\]

Spin down rate:

\[
\frac{d\Omega}{dt} = \frac{C}{6} (\Omega_{in}^{-6} - Ct)^{-7/6}.
\]

\[
t_c = \frac{1}{C} (\Omega_{in}^{-6} - \Omega_c^{-6}).
\]
THEORETICAL CALCULATIONS

\[ I(R_c) = \int_0^{R_c} \left[ \frac{\epsilon(r)}{\text{MeV fm}^{-3}} \right] \left( \frac{r}{\text{km}} \right)^6 d\left( \frac{r}{\text{km}} \right). \]

\[ \tau_{GR} = -0.7429 \left[ \frac{R}{\text{km}} \right]^9 \left[ \frac{1M_\odot}{M} \right]^3 [I(R_c)]^{-1}(s), \]

\[ \tau_{ee} = 0.1446 \times 10^8 \left[ \frac{R}{\text{km}} \right]^{3/4} \left[ \frac{1M_\odot}{M} \right]^{1/4} \left[ \frac{\text{km}}{R_c} \right]^6 \times \left[ \frac{\text{g cm}^{-3}}{\rho_t} \right]^{1/2} \left[ \frac{\text{MeV fm}^{-3}}{\epsilon_t} \right] [I(R_c)](s), \]

\[ \tau_{nn} = 19 \times 10^8 \left[ \frac{R}{\text{km}} \right]^{3/4} \left[ \frac{1M_\odot}{M} \right]^{1/4} \left[ \frac{\text{km}}{R_c} \right]^6 \times \left[ \frac{\text{g cm}^{-3}}{\rho_t} \right]^{5/8} \left[ \frac{\text{MeV fm}^{-3}}{\epsilon_t} \right] [I(R_c)](s), \]

\[ \Omega_c \sim \frac{R_c^{12/11}}{[I(R_c)]^{4/11}} \rho_t^{3/11} \]
FIG. 4. Plots of fiducial timescales with gravitational mass of neutron stars with DDM3Y EoS.

FIG. 6. Plots of critical temperature versus mass.
FIG. 8. Plots of time evolution of frequencies.


FIG. 10. Plots of spin-down rates vs frequencies.
TABLE IV. Spin frequencies and core temperatures (measurements and upper limits) of observed low-mass x-ray binaries (LMXBs) and millisecond radio pulsars (MSRP) [82].

<table>
<thead>
<tr>
<th>Source</th>
<th>$\nu$ (Hz)</th>
<th>$T_{\text{core}}$ (10^8 K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aql X-1</td>
<td>550</td>
<td>1.08</td>
</tr>
<tr>
<td>4U 1608-52</td>
<td>620</td>
<td>4.55</td>
</tr>
<tr>
<td>KS 1731-260</td>
<td>526</td>
<td>0.42</td>
</tr>
<tr>
<td>MXB 1659-298</td>
<td>556</td>
<td>0.31</td>
</tr>
<tr>
<td>SAX J1748.9-2021</td>
<td>442</td>
<td>0.89</td>
</tr>
<tr>
<td>IGR 00291+5934</td>
<td>599</td>
<td>0.54</td>
</tr>
<tr>
<td>SAX J1808.4-3658</td>
<td>401</td>
<td>&lt;0.11</td>
</tr>
<tr>
<td>XTE J1751-305</td>
<td>435</td>
<td>&lt;0.54</td>
</tr>
<tr>
<td>XTE J0929-314</td>
<td>185</td>
<td>&lt;0.26</td>
</tr>
<tr>
<td>XTE J1807-294</td>
<td>190</td>
<td>&lt;0.27</td>
</tr>
<tr>
<td>XTE J1814-338</td>
<td>314</td>
<td>&lt;0.51</td>
</tr>
<tr>
<td>EXO 0748-676</td>
<td>552</td>
<td>1.58</td>
</tr>
<tr>
<td>HETE J1900.1-2455</td>
<td>377</td>
<td>&lt;0.33</td>
</tr>
<tr>
<td>IGR J17191-2821</td>
<td>294</td>
<td>&lt;0.60</td>
</tr>
<tr>
<td>IGR J17511-3057</td>
<td>245</td>
<td>&lt;1.10</td>
</tr>
<tr>
<td>SAX J1750.8-2900</td>
<td>601</td>
<td>3.38</td>
</tr>
<tr>
<td>NGC 6440 X-2</td>
<td>205</td>
<td>&lt;0.12</td>
</tr>
<tr>
<td>Swift J1756-2508</td>
<td>182</td>
<td>&lt;0.78</td>
</tr>
<tr>
<td>Swift J1749.4-2807</td>
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<td>&lt;1.61</td>
</tr>
<tr>
<td>J2124-3358</td>
<td>203</td>
<td>&lt;0.17</td>
</tr>
<tr>
<td>J0030+0451</td>
<td>205</td>
<td>&lt;0.70</td>
</tr>
</tbody>
</table>

FIG. 7. Plots of critical frequency with temperature for different masses of neutron stars. The square dots represent observational data.
CONCLUSIONS

- Massive hot young neutron stars are more susceptible to gravitational radiation due to r-mode instability.

- According to our model of the DDM3Y EoS, none of the observed MSRP s and LMXBs lie in the r-mode instability region which is consistent with the lack of detection of gravitational waves from isolated neutron stars.

- The Symmetry Energy Slope Parameter L of the DDM3Y EoS is 45.1066 MeV. According to Vidana (I. Vidaña, Phys. Rev. C 85, 045808 (2012)) if we consider the shear viscosity along the crust-core boundary layer as the dominant dissipation mechanism, then the r-mode instability window is increased to lower values of L.

THANK YOU